

RiemCirc: A Generator of Nodes and Weights for Riemann Integration on the Circle

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RiemCirc is a C++ program which allocates points inside the unit circle for numerical quadrature on the circle, aiming at homogeneous equidistant distribution. The weights of the quadrature rule are computed by the area of the tiles that surround these nodes. The shapes of the areas are polygonal, defined by Voronoi tessellation.

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I. SCOPE

A. Quadrature

Numerical integration of functions defined on the unit circle is generally discussed with a wish to preserve the (or some form of subgroup) symmetry of the circle while choosing a set of nodes and weights of sampling the area [1–3]. The integral is canonically replaced by a finite Riemann sum over N nodes with radial and azimuthal coordinates r_j and φ_j ,

$$\int f(r, \varphi) d^2r \approx \sum_{j=1}^N w_j f(r_j, \varphi_j). \quad (1)$$

The standard procedure is to split the integral into a product of integrals over r and φ and to select separate one-dimensional Gaussian integration formulas and associated nodal weights [4–6]. (We will not be interested in integrations along the perimeter of the circle which typically arises in integrations in the complex plane [7, 8].)

The disadvantage of the product representation is a clumpy structure of the nodes in the circle if N is not large. The program that is presented here ignores aspects of symmetry conservation and attempts to distribute N points well-balanced in the sense that the weights w_j have approximately equal values, in fact, minimizing their variance.

B. Voronoi Tessellation

The idea is that the distances between the nodes are equilibrated by starting from a randomized set of locations, then moving the nodes iteratively as if some repelling force was adjusting their places. This is put into concrete by (i) subdividing the area inside the circle into tiles around the nodes akin to a Voronoi tessellation, and (ii) moving the individual nodes to the center-of-mass of

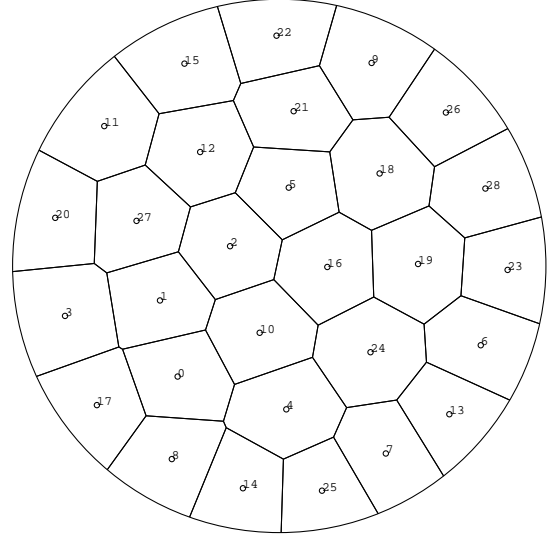


FIG. 1. Illustration of a result on 29 nodes.

their tile, and iterating these two steps a few times to achieve some stable configuration.

Examples of nodes and associated tiles on $N = 29$ nodes are plotted in Figures 1 and 2. Since 29 is a prime number, the usual factorization method would need to align all points on a simple ring around the center or place one at the center and distribute the others in 7 or 4 annuli.

II. IMPLEMENTATION

A. Specifications

The generic aspects of the algorithm are

- The definition of weights w_j as the area of tiles surrounding N fixed points. The sum is $\sum_j w_j = \pi$, the area of the unit circle.
- The specification of these tiles via iterated splitting of the full area with line sections that cut mid-way between any pair of two points. This is the Voronoi tessellation, and the definition of Brillouin zones in solid state physics.

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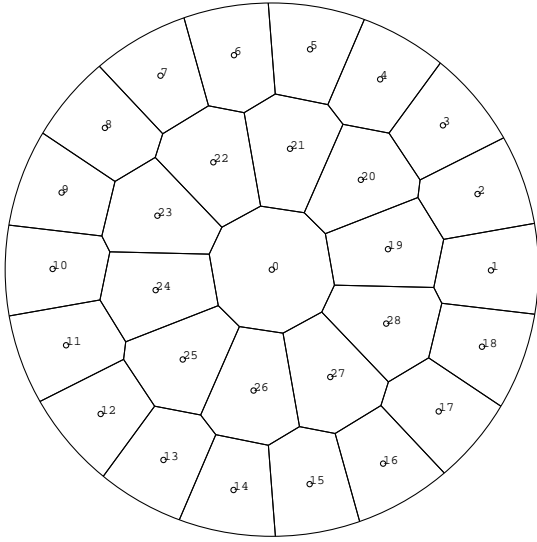


FIG. 2. Illustration of a “centered” result on 29 nodes.

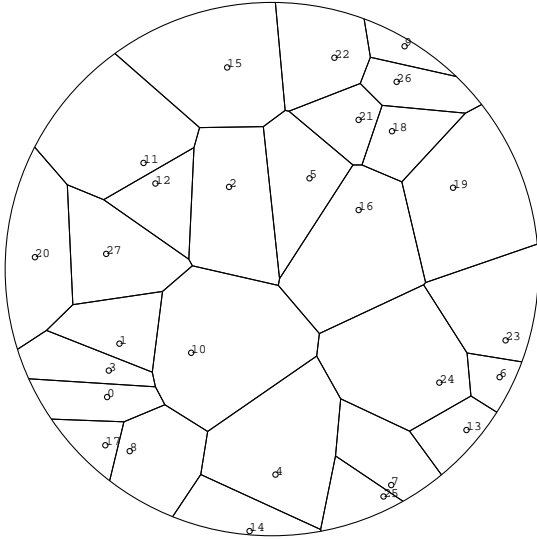


FIG. 3. Illustration of an initial, randomized placement of 28 nodes without (or prior to) iteration.

- The re-definition of points via tiles as their center-of-mass coordinates (centroidal tessellation [9, 10]).

The computation of polygonal areas poses no further problem, and is done by adding the triangular areas of the sides as viewed from their centers. The points near the rim of the circle are individually detected and the sub-area of their spherical caps is included in their weight.

B. Modes of Use

The program has a mode of “fix-point” use in which the number and coordinates of nodes are preassigned and read in from an ASCII file. The program just subdivides

the region and computes the weights (areas) if no relocation loop is requested.

The other invocation uses a random-number generator to initialize point positions in the circle area. With a variable seed of the random-number generator, different point sets can be initiated.

In both cases, a number (defaulting to 0) of iterations of moving the points to the tile centers and adjusting the polygons may be specified. This achieves “self-consistent” stable results with small variances in the weights after a modest number of loops. If that loop count is kept at zero, the points are left frozen at their initial positions, and shapes like in Figure 3 with a larger spread of tile areas result.

The generic output is a list of the two Cartesian (optionally two circular) coordinates and one weight of each of the N points in ASCII format, optionally bracketed for inclusion as arrays in other programming codes.

As a visual aid, PostScript images showing the nodes and tiles may be created, as seen in the figures above.

III. SUMMARY

We have implemented a strategy of dividing the interior of the unit circle into distinct non-overlapping tiles defined by halving the distance between neighboring abscissa (nodal) points. These tiles define weights and nodes for Riemann integration over the unit circle. The C++ source code is made available in the ancillary files.

Appendix A: Installation

The program is compiled by moving to the source directory and calling the C++ compiler with

make

and moving the executable **RiemCirc** to a directory where the operating system will find it—depending on definitions of environment variables.

The Unix man-page **RiemCirc.1** may also be moved to the subdirectory **man/man1** of the standard folder locations or be read with

nroff -man RiemCirc.1 | more

In case that **doxygen** is available, one may also call **make doc** to generate a PDF file with the API.

Appendix B: Synopsis

A summary of the two versions of calling the program is

```
RiemCirc [-v] [-C] [-i file.asc] [-p file.ps] [-a] [-l Nrelo]
RiemCirc [-v] [-C] [-p file.ps] [-a] [-l Nrelo] [-r seed]
N
```

Brackets surround optional arguments.

The first variant reads the initial positions from *file.asc*, whereas the second variant chooses them randomly (if the *seed* is specified) or in annuli (if the seed is not used).

The option `-v` causes the variance of the areas to be reported.

The option `-C` means that C-style curly parentheses will be added to the output.

The option `-p` lets the program generate the PostScript file *file.ps* in the style seen in the manuscript.

The option `-a` causes an output of point locations in r and φ coordinates rather than the default x and y coordinates.

The option `-l` lets the program run *Nrelo* relocation and re-tessellation loops on the initial placements before creating the outputs.

The option `-r` places the N points initially at random inside the unit circle. The result will look more like Figure 1 than the ring oriented Figure 2 after some iterations.

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